WAMO 2

Alex Yang, Bruce Shu, James Yang, Kaiyuan Mao, Laura Wang

Patrick Sun, Ryan Chen, Ryan Tang, Wesley Wu

1. Find the number of factors of the smallest positive integer n such that $n \cdot 20!$ is a perfect square.

2. Alex has 20 green marbles, 15 blue marbles, and 10 red marbles. He randomly places each of his marbles into one of three boxes colored red, blue, and green with a $\frac{1}{3}$ probability of placing each marble into any of the 3 boxes. What is the expected number of balls whose color matches the color of the box that it is inside?

3. A "fun coin", is a coin such that the probability to land on heads, tails, or the side is all equal. If Biff flips the coin 6 times, the probability that at some point after the first flip, Biff has flipped the coin an equal amount on its head, tail, and side can be expressed as $\frac{a}{b}$, where a and b are relatively prime. Find $a + b$.

4. Let $f(x) = x^2 - 24x + 46$ and $g(x) = f(f(f(f(x))))$. Find $g(20) + g(2) - g(4)$.

5. A circle with radius 2 is in the center of a rectangle with dimensions 10 by 20. The circle rotates at a speed of one revolution per second. A flashlight is on the edge of the circle. Every $\frac{1}{16}$ of a second, the flashlight toggles between on and off. Given that the flashlight starts off facing left and is on, the total area of the region that experiences light can be expressed as $a - b\pi$. Find $a + b$.

6. Real numbers a, b, c satisfy $a^2 - b^2 = 1 + \sqrt{2}$, $b^2 - c^2 = 1 - \sqrt{2}$, and $c + a = \sqrt{2} + \sqrt{5}$. Find the value of $a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2$.

7. Fourteen circles are drawn, with 12 of the congruent circles externally tangent to the 13th circle, which is inside the twelve other circles. The 14th circle is internally tangent to the 12 circles, "encapsulating" the 13 other circles. If the radius of each of the 12 congruent circles is 34, the sum of the diameters of the other 2 circles can be expressed as $a\sqrt{b} + c\sqrt{d}$, where b and d are square-free. Find $a + b + c + d$.

8. Acute triangle ABC with incenter I and $AB = 13, AC = 20$, and $BC = 21$. Line AI is doubled in length past I to point M. The perpendicular from M

to BC intersects BC at P and the incircle at Q and R, with Q on PR. Find $MP \cdot MQ \cdot MR$.

9. Rectangle $WAMO$ has $WA = 34$ and $AM = 14$. Square $RDFE$ is drawn with RD on WA such that the combined area of square $RDFE$ and trapezoid $EFMO$ is minimized. Compute the side length of $RDFE$.

10. Given that $24 \log_a x \log_b x \log_c x \log_d x = \log_a x \log_c x + \log_a x \log_d x + \log_b x \log_c x + \log_b x \log_d x$, $abcd = x^{14}$, and the minimum $a+b+c+d$ is equal to 1464, compute x.

11. Let S be the sum of the squares of the factors of 1434. Compute the sum of the digits of $\sqrt{S}$.

12. The parabola $y = \frac{1}{10}x^2$ is drawn, and a circle with radius 13 is drawn tangent to the parabola at two points. Another larger circle is drawn tangent to the parabola at two points and the first circle. What is the radius of the larger circle?

13. Ryan has a list of the first 100 positive square numbers. He then randomly shuffles the list and concatenates all the numbers together in that order. The expected remainder of this number when divided by 90 can be expressed as $\frac{a}{b}$, where a and b are coprime. Find $a + b$.

14. How many integers x satisfy $0 \leq x \leq 314$ and $\sin(x)\sin(x+3) > 0$?

15. In quadrilateral $ABCD$, the midpoints of AB, BC, CD, and DA are M, N, O, and P, respectively. Furthermore, let the midpoint of MN be Q. Given that $MO = 4\sqrt{10}$, $NP = 2\sqrt{29}$, $BD = 2\sqrt{113}$, $DQ = 11\sqrt{2}$, and $AD = CD$, the perimeter of $ABCD$ can be expressed as $\sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $a + b + c$.

16. Uh oh! While writing out Pascal's Triangle, Yaptrick forgot the recursive property for the triangle, and instead of adding the two numbers above a space to create a new number, he added an extra 1 to the sum. The first few rows of his triangle look like this:

1	(Row 0)
2 2	(Row 1)
3 5 3	(Row 2)
4 9 9 4	(Row 3)

He defined $\left[{m \atop n}\right]$ as the $(n+1)^{\text{th}}$ number in Row m of his triangle. For example, $\left[{3 \atop 2}\right] = 9$ and $\left[{2 \atop 1}\right] = 5$. Find the remainder when $\left[{20 \atop 10}\right]$ is divided by 100.

17. In acute triangle ABC, the orthocenter is H. The midpoint of AH is M. If $AM^2 = \frac{9}{5}$, $\left(\frac{CM}{BM}\right)^2 = \frac{83}{58}$, and $BC = 9$, find the value of $AB + AC$.

18. Find the remainder when the sum of the digits of $\lfloor \frac{10^{10^{100}}}{1001} \rfloor$ is divided by 1001.

19. Let a, b, and c be the roots of $x^3 - 11x^2 + 14x - 2$. Let $x = \sqrt{a^2 + b^2 + ab}$, $y = \sqrt{b^2 + c^2 + bc}$, and $z = \sqrt{c^2 + a^2 + ca}$. Compute $(x + y + z)(-x + y + z)(x - y + z)(x + y - z)$.

20. Let f be a monic quadratic with integer coefficients of the form $f(x) = x^2 - ax + b$ with $a, b > 0$. With proof, find all functions f such that all the roots of $f(f(x))$ are integers.